DRAWING GRAPHS ON SURFACES

KURATOWSKI'S THEOREM - A CHARACTERISATION OF PLANAR GRAPHS

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Abstract

Graph theory is the study of graphs, a type of mathematical structure consisting of points, called vertices, that are connected by lines, called edges. Kasimierz Kuratowki (1896-1980) was a Polish mathematician known for his theorem in this area of Mathematics, namely that one only needs to look at the structure of a graph to decide whether it can be drawn in the plane without intersecting edges. This article shows several properties of these graphs, provides a detailed proof of the characterisation and takes a glance at similar results on other surfaces, focusing on the sphere and the torus.

1 Introduction

In 1930, a Polish mathematician by the name of Kasimierz Kuratowski published an article entitled "Sur le problème des courbes gauches en topologie" [7]. The aforementioned paper established a link between the fields of graph theory and topology by proving that one only needs to look at the structure of a graph to determine whether it can be drawn in the plane without intersections [7]. This is the idea behind Kuratowski's Theorem, which has since then become the most frequently cited result in graph theory [2]. Similar results were proved independently and around the same time by Pontryagin in the Soviet Union as well as Frink and Smith in the United States [6]. However, their results were never published since Kuratowski's paper was already in press. To fully grasp the significance of Kuratowski's Theorem, we begin with the formal definition of a graph.

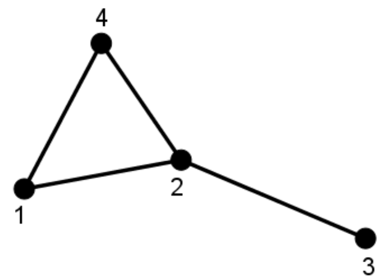
Definition 1.1.1. [8] A *graph* G is a pair of sets (V,E) where V is nonempty and E is a set of 2-element subsets of V. The elements in the set V are called vertices and the sets in E are called edges. For an edge $e = \{u,v\} \in E$, the vertices u and v are called the ends or endpoints of e.

Remark. To avoid confusion whenever we consider multiple graphs, we denote the set of vertices of a graph G by $V(G)$ and the set of edges by $E(G)$

While abstract in nature, graphs have a wide variety of applications in electrical engineering, computer science and physics. For more information about applications, see [5].

To make the notion of graph more concrete, we can represent an abstract graph G in the plane by depicting vertices by dots and edges by lines. We will call such a drawing an *embedding* of the graph G.

Example. Consider the graph $G = (V,E)$ with $V = \{1,2,3,4\}$ and $E = \{\{1,2\},\{1,4\},\{2,3\},\{2,4\}\}$. Then, we can represent G in the plane as follows:

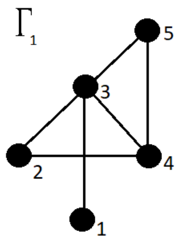


To further characterize the structure of a graph, we need additional terminology to refer to vertices that are connected by an edge.

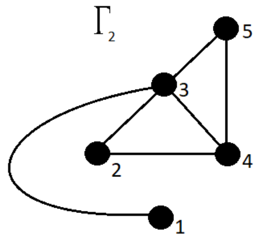
Definition 1.1.2. [8] Two vertices u and v of a graph G are called *adjacent* in G if there exists an edge of G with u and v as its ends. The vertices u and v are said to be *neighbours* in G.

Definition 1.1.3. [5] A *planar embedding* of a graph G is an embedding of G in which no two edges intersect and no two vertices coincide. We will call the graph G *planar* if there exists such an embedding of G.

Example. Consider the graph G where Γ_1 and Γ_2 are two embeddings of the graph G. Note that G is a planar graph since it has a planar embedding Γ_2.



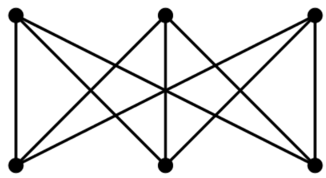
Not a planar embedding of G



A planar embedding of G

It is interesting to note that some graphs do not have a planar embedding. In other words, these graphs can never be drawn in the plane without intersections. We will refer to these graphs as *nonplanar graphs*. We will consider two such examples.

The first originates from a famous puzzle called the "*Three utilities problem*". The task is to connect three cottages to three companies in such a way that none of the connections intersect each other without using the third dimension. We can restate the puzzle using notions of graph theory by asking for a planar embedding of the graph below, which is known as the complete bipartite graph $K_{3,3}$.

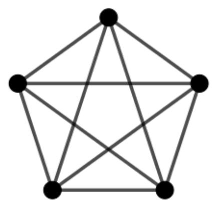


An embedding of $K_{3,3}$

Definition 1.1.4. [9] *The complete bipartite graph* $K_{3,3}$ is the graph with vertices $\{v_1, v_2, v_3\} \cup \{u_1, u_2, u_3\}$ and all 2-sets $\{v_i, u_j\}$ as its edges, where $1 \le i \le 3$ and $1 \le j \le 3$.

The second nonplanar graph that we consider is the graph with 5 vertices where each vertex is connected to each other vertex by a unique edge, which is known as the complete graph K_5.

Definition 1.1.5. [9] The *complete graph* K_5 is the graph with 5 vertices in which any two vertices are adjacent.

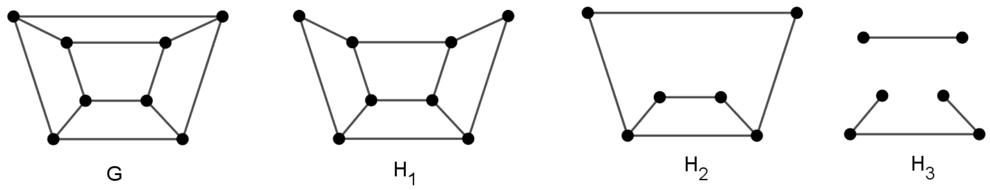


An embedding of K_5

It is intuitively clear that adding vertices and/or edges to a nonplanar graph, still results in a nonplanar graph. Hence, in order to determine whether some graph G is nonplanar, it suffices to find a nonplanar graph contained in G. We will now formalize what it means for a graph to be contained in another graph.

Definition 1.1.6. [9] The graph $H = (V', E')$ is a *subgraph* of the graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Example. Consider the graphs G, H_1, H_2 and H_3 embedded in the plane as follows

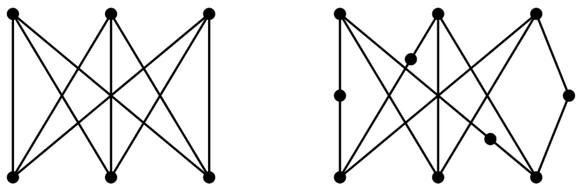


Examples of subgraphs of G

We see that the graphs H_1, H_2 and H_3 are subgraphs of G as $V(H_i) \subseteq V(G)$ and $E(H_i) \subseteq E(G)$ for $i \in \{1,2,3\}$.

We can also generate new graphs from some existing graph G by placing a vertex with two neighbours an edge of G. This process is called subdividing an edge in G.

Definition 1.1.7. [15] The graph H is a *subdivision* of the graph G if H can be constructed from G by applying successive edge subdivisions.



Graph G and graph H respectively

We are now ready to give a complete statement of Kuratowski's Theorem.

Theorem 1.1.8 (Kuratowski's Theorem)**.** [15] A graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5 as a subgraph.

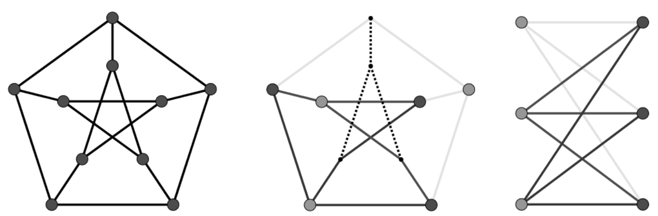
We can restate the theorem more clearly by introducing the following definition.

Definition 1.1.9. [9] The graph H is a Kuratowski subgraph of the graph G if H is a subgraph of G and H is a subdivision of $K_{3,3}$ or K_5.

Hence, the theorem now reads:

Theorem 1.1.10 (Kuratowski's Theorem)**.** [9] A graph is planar if and only if it does not contain a Kuratowski subgraph.

We might consider an example of the theorem to prove that the *Peterson graph*, exposed in the following drawing, is nonplanar. To do so, one simply needs to observe that the graph contains a Kuratowski subgraph, and more specifically it contains a subdivision of $K_{3,3}$ as a subdivision.



The Peterson graph (Leftmost graph), $K_{3,3}$ (Middle and rightmost graph)

The above statement provides necessary and sufficient conditions for a graph to be planar. Hence, any graph containing a subdivision of $K_{3,3}$ or K_5 must necessarily be nonplanar. Conversely, we also have that subdivisions of $K_{3,3}$ and K_5 are the only graphs which can cause a graph G to be nonplanar. More precisely, if a graph G is nonplanar, then we can find a subdivision of $K_{3,3}$ or K_5 contained in G.

It is interesting to note that a similar theorem was proved independently and around the same time by an Austrian-American mathematician called Karl Menger [6]. More precisely, he showed that a graph where all vertices have degree 3 is planar if and only if it does not contain a subdivision of $K_{3,3}$ as a subgraph.

The main objective of our paper is to give a complete proof of Kuratowski's Theorem [15]. We also consider the question whether the theorem generalizes to other surfaces [10]. Section 2 contains an investigation into the properties of planar graphs [8]. In particular, we state and prove Euler's characteristic formula, along with other inequalities, which gives a relation between the number of vertices, the number of edges and the number of faces of planar graphs [14]. These results are then used to prove that both $K_{3,3}$ and K_5 are nonplanar. Section 3 aims to prove the statement of Kuratowski's theorem [12]. We do this by first showing that the result holds for a special class of graphs. This result is known as Tutte's theorem, which is proved by induction on the number of vertices. We use this result to derive a contradiction under the assumption that there exists a graph which is nonplanar and does not contain a Kuratowski subgraph [9]. Section 4 deals with the question whether Kuratowski's theorem holds on other surfaces. More precisely, we consider the sphere and the torus. We also see that the non-embeddable graphs on these surfaces differ from the ones in the plane [10].

2 Elementary nonplanarities

Before discussing properties of planar graphs, we introduce some notation which will make the proof easier to read and more understandable.

Notation. For a graph G, we denote by $V(G)$,$E(G)$ and $F(G)$ respectively the sets of vertices, edges and faces of G. We then denote by $v(G)$,$e(G)$ and $f(G)$ the cardinalities, the number of elements, of the respective sets. When no confusion is possible, we may omit the arguments and denote the sets and quantities simply by V, E, F, v, e and f.

We do already know some links between these different notations. For instance, we know that $E(G)$ is a 2-element subset of $V(G)$. In this section we will be more interested in connecting the cardinalities of the sets, by equalities and inequalities, for special graphs, namely planar graphs. Once the inequalities established we will be able to easily prove the nonplanarity of specially chosen graphs.

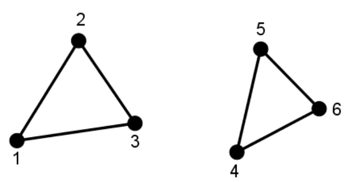
2.1 Euler's Characteristic Formula

Kuratowski's theorem states that all graphs containing a subdivision of K_5 or $K_{3,3}$ are nonplanar. Therefore, we start proving Kuratowski's theorem by showing that K_5 and $K_{3,3}$ are nonplanar. To prove this, we introduce Euler's characteristic formula, an equality which relates v, e and f. This, together with an inequality between e and f gives the needed result to proof that K_5 and $K_{3,3}$ are nonplanar.

We note that Euler's characteristic formula is only true for connected planar graphs. Hence, we introduce the notion of *components* and use it to define a *connected* graph. We also define the notion of a *path*.

Definition 2.1.1. [5] Subgraphs $G[V_1]$, $G[V_2]$,..., $G[V_\omega]$ are *components* of G where V_1, V_2,..., V_ω are nonempty sets contained in V such that every 2 vertices v, u are connected by an edge if and only if they belong to the same subset v_i.

Example. The following graph contains 2 components as none of the vertices 1, 2, 3 are connected to vertices 4, 5 or 6.



Definition 2.1.2. [8] A graph G is said to be *connected* if it consists of only one component. A graph with multiple components is *disconnected*.

The graph in the example above is disconnected as it has 2 components.

Throughout the rest of the paper a lot of proofs shall rely on the principle of mathematical induction. We will thus need to introduce transformations which reduce the structure of a graph.

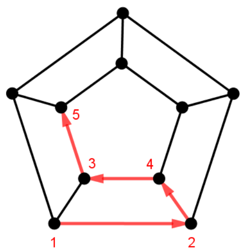
Definition 2.1.3. For $S \subseteq V(G)$, we denote by $G - S$ the subgraph of G obtained by deleting from G the vertices in S and all edge adjacent to them. We call the resulting graph a *vertex deleted subgraph* of a graph G.

Definition 2.1.4. For $S \subseteq E(G)$, we denote by $G - S$ the subgraph of G obtained by deleting from G the edges in S. We call the resulting graph a *edge deleted subgraph* of a graph G.

We will also want to introduce the notion of a path on a graph. This will come in very useful later on during the proofs.

Definition 2.1.5. A *path* is a set of vertices $w = v_0 v_1 v_2 v_3 \cdots v_n$ such that $v_i \neq v_j$ for $(i \neq j)$. And for any $0 \leq i \leq n-1$, we have that $e = v_i v_{i+1}$ exist.

Intuitively, a *path* is a route from 1 vertex to another vertex by traversing over the edges in the graph. Each vertex can only be used once.



Now that we have introduced the notion of connected graphs, we can prove Euler's characteristic formula, which states that for any planar graph some linear function links the number of vertices, the number of edges and the number of faces.

Theorem 2.1.6 (Euler's Characteristic Formula)**.** [3], [14] For any connected planar graph G, the following holds

$$v - e + f = 2$$

Proof.

In the following proof, we use a generalization of graphs, we namely allow for multiple edges between one pair of vertices. If the formula holds for this generalization, it must also hold for simple graphs, without multiple edges between pairs of vertices.

To prove this result we apply induction on the number of vertices in a graph G, $v(G)$.

Base case.
For the base case, we take G to be an isolated vertex placed in the plane. Then $v(G) = 1$, $e(G) = 0$ and $f(G) = 1$. We may thus conclude

$$v(G) + e(G) + f(G) = 1 - 0 + 1 = 2$$

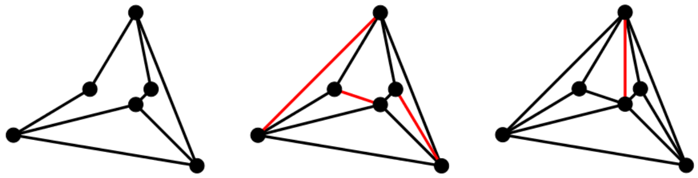
Thus Euler's formula holds for this graph.

Induction Step.
Take any connected planar graph G with n vertices and embed G in the plane. Define the *triangulation algorithm* as follows:

1. Identify a face of G that is adjacent to 4 or more edges if possible. If this is not possible, terminate the algorithm.
2. The identified face has at least 4 vertices on its boundary, meaning that we can find 2 vertices that are not next to each other on the boundary of the face. Choose such 2 vertices.
3. Connect these 2 vertices by an edge going trough the aforementioned face. Note that by assuming that a pair of vertices can be connected by multiple edges, this is always possible.
4. Return to step 1.

We call the graph after termination of the algorithm G'.

Example. To clarify the triangulation process, we look at the following example. In the first picture, we iterate over the algorithm 3 times to add edges in 3 of the faces. The next picture shows the last iteration, after which the algorithm terminates as all faces have exactly 3 edges on their boundary.



It is important to note that this algorithm terminates. Denote by $G^{(i)}$ the graph after i iterations. To prove the termination of the algorithm, define the function $\lambda\colon F(G) \mapsto \mathbb{N}$ such that

$$\lambda(f) = \begin{cases} \#\{\text{vertices on the boundary of } f\} - 3 & \text{if this number is positive} \\ 0 & \text{else} \end{cases}$$

Next, define

$$\Lambda^{(i)} = \sum_{F(G^{(i)})} \lambda(f)$$

At every iteration of the algorithm, the algorithm either terminates or a face $f^{(i)}$ with n vertices on its boundary gets split into 2 faces, $f_1^{(i+1)}$ and $f_2^{(i+1)}$. These two faces have respectively a and $n-a+2$ vertices on their boundary, for some integer value $a \geq 3$. Note that $n-a+2 \geq 3$ as well. This means that

$$\begin{aligned} \Lambda^{(i+1)} &= \sum_{F(G^{(i+1)})} \lambda(f) \\ &= \left(\sum_{F(G^{(i)})} \lambda(f)\right) - \lambda(f^{(i)}) + \lambda(f_1^{(i+1)}) + \lambda(f_2^{(i+1)}) \\ &= \Lambda^{(i)} - (n-3) + (a-3) + (n-a+2-3) \\ &= \Lambda^{(i)} - 1 \\ &< \Lambda^{(i)} \end{aligned}$$

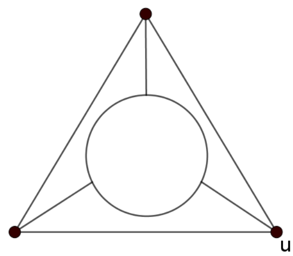
Since $\Lambda^{(i)}$ is a positive integer, the sequence $(\Lambda^{(i)})$ cannot decrease indefinitely, so the algorithm must terminate eventually.

At every iteration of the algorithm, we add an edge to a face which divides the face in 2, thus adding a face. Hence we increase $f(G)$ by one and $e(G)$ by one, thus obtaining

$$v(G) - e(G) + f(G) = v(G') - e(G') + f(G')$$

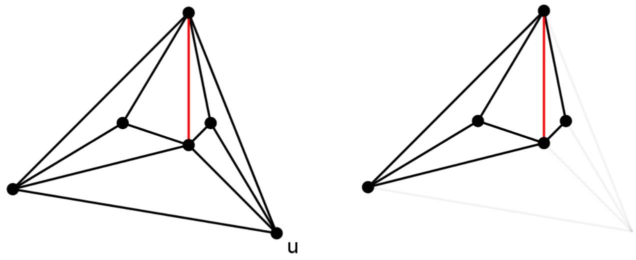
Now we identify 1 of the 3 vertices, u, on the boundary of the unbounded face. Note that $G' - \{u\}$ has less vertices than G, since it has less vertices than G', which has the same number of vertices as G. Additionally, observe that $G' - \{u\}$ must be connected, due to the nature of the triangulation. This means that the induction hypothesis implies $G' - \{u\}$ respects the Euler's Characteristic Formula.

Example. The picture below is an overview of the last layer after the triangulation process, where the circle in the middle represents some connected graph. Observe that $G' - \{u\}$ is connected.



Removing u decreases $v(G')$ by 1, $e(G')$ by $d(u)$ (we delete all edges connected to u) and $f(G')$ by $d(u)-1$.

Example. In the graph below, we remove vertex u where each face has exactly 3 edges as boundary. Indeed, we remove $d(u) = 4$ edges and $d(u) - 1 = 3$ faces.



On the left the original graph and on the right the graph after removing vertex u

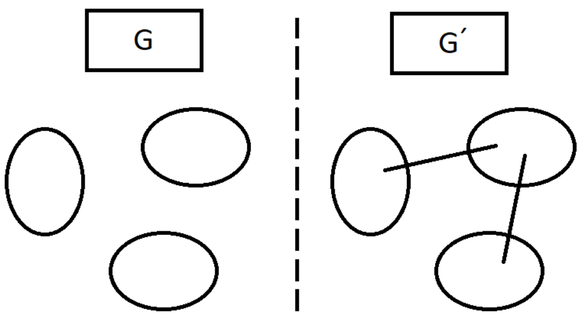
Now we combine the fact that both the triangulation process as well as removing u does not change Euler's characteristic formula. We conclude:

$$\begin{aligned}
2 &= v(G' - \{u\}) - e(G' - \{u\}) + f(G' - \{u\}) \\
&= \left[v(G' - \{u\}) + 1\right] - \left[e(G' - \{u\}) + d(u)\right] + \left[f(G' - \{u\}) + d(u) - 1\right] \\
&= v(G') - e(G') + f(G') \\
&= v(G) - e(G) + f(G)
\end{aligned}$$

Hence, Euler‘s characteristic formula also holds for G which finishes the induction proof.

□

Note that Euler's characteristic formula does not hold for disconnected graphs. To make this explicit, consider a planar graph G with k components. Then, by adding $k-1$ cut edges to G, we obtain a connected planar graph G' while neither changing the number of faces nor vertices. Hence, the equality fails for the original disconnected graph.



2.2 Applications of Euler's Characteristic Formula

We use Euler's characteristic formula to prove the nonplanarity of $K_{3,3}$ and K_5. However, we first need one more relation between the number of edges and faces of vertices for a connected planar graph before we get to the desired result.

Lemma 2.2.1. [5] For any planar connected graph, the following inequality between edges and faces hold

$$2e \geq 3f$$

Proof.

To prove this, we introduce a δ function

$$\delta(e,f) = \begin{cases} 1 & \text{if } e \text{ is on the boundary of } f \\ 0 & \text{else} \end{cases}$$

We know that any edge is at most on the boundary of 2 different faces. Hence, evaluating $\delta(e,f)$ at a fixed edge while summing over all faces yields a value of 2 or less:

$$\sum_F \delta(e,f) \leq 2$$

Following this procedure for all edges hence yields the following:

$$\sum_E \sum_F \delta(e,f) \leq 2e$$

We know that any face is bounded by at least 3 edges. Therefore, evaluating $\delta(e,f)$ at a fixed face, whilst summing over the edges yields a value larger than 3. Following this procedure for any face yields the following result:

$$\sum_F \sum_E \delta(e,f) \geq 3f$$

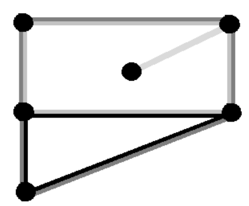
We combine these cases to get the desired result:

$$2e \geq \sum_E \sum_F \delta(e,f) = \sum_F \sum_E \delta(e,f) \geq 3f$$

$$\implies 2e \geq 3f$$

□

Remark. We can intuitively assess the truth of this statement by looking at the graph below. Each edge has 2 colours representing the 2 faces it is adjacent to. If an edge is adjacent to 1 face, we do not colour it. Note that each face accounts for at least 3 coloured edges and each coloured edge has at least 2 colours. Combining this also yields $2e \geq 3f$.



Substituting this result in Euler's formula gives a relation between the number of vertices and edges for any planar graph, which is the final result that is used to proof the nonplanarity of $K_{3,3}$ and K_5.

Lemma 2.2.2. [5] For any planar graph, the following relation holds

$$e \leq 3v - 6$$

Proof. If we substitute $2 \geq 3$, obtained in lemma 2.2.1, into Euler's characteristic formula, obtained in theorem 2.1.6, we get

$$2e \geq 3(2 + e - v)$$

Rewriting gives the desired result

$$e \leq 3v - 6$$

□

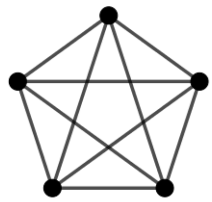
This inequality allows us to prove that K_5, the graph with 5 vertices all connected to each other, is nonplanar.

Corollary 2.2.3. [5] The graph K_5 is nonplanar.

Proof. K_5 has 10 edges and 5 vertices. For K_5 to be planar, the following equation derived in lemma 2.2.2 should hold

$$10 \leq 3 \cdot 5 - 6 = 9$$

which is not the case. Therefore, K_5 cannot be planar.



□

A similar result to lemma 2.2.2 may be obtained for bipartite graphs. Before doing so, we introduce the notion of a bipartite graph.

Definition 2.2.4. [8] A graph G is a *bipartite graph* whenever the set of vertices can be partitioned into 2 disjoint sets such that no two vertices in the same set are connected by an edge.

Lemma 2.2.5. [5] For any planar bipartite graph with $v \geq 4$ edges, the following inequality holds

$$e \leq 2v - 4$$

Proof.

The proof is analogous to the proof of lemma 2.2.1 and 2.2.2.

First, observe that the following holds

$$e \leq 2f$$

To prove this inequality, define a δ function

$$\delta(e, f) = \begin{cases} 1 & \text{if } e \text{ is on the boundary of } f \\ 0 & \text{else} \end{cases}$$

We know that any edge is at most on the boundary of 2 different faces. Hence, evaluating $\delta(e,f)$ at a fixed edge while summing over all faces yields a value of less than 2:

$$\sum_F \delta(e,f) \leq 2$$

Following this procedure for all edges hence yields the following result

$$\sum_E \sum_F \delta(e,f) \leq 2e$$

We know that any face is bounded by at least 4 edges. This is true since a face in any graph is bounded by at least 3 faces, but the bipartite nature of G forces a face to not be bounded by exactly 3 edges, which would imply 2 vertices in the same part to be connected. Therefore, evaluating $\delta(e,f)$ at a fixed face, whilst summing over the edges yields a value larger than 3. Following this procedure for any face yields the following result:

$$\sum_F \sum_E \delta(e,f) \geq 4f$$

We combine these cases to get the desired result:

$$2e \geq \sum_E \sum_F \delta(e,f) = \sum_F \sum_E \delta(e,f) \geq 3f \implies 2e \geq 4f$$

$$\implies e \geq 2f$$

Plugging this into the Euler Characteristic Formula, we obtain

$$2 = v - e + f \leq v - e + e/2$$

$$\implies e \leq 2v - 4$$

as wanted.

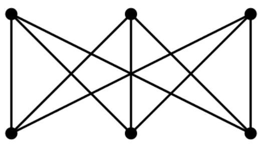
□

Corollary 2.2.6. [5] The graph $K_{3,3}$ is non-planar.

Proof. $K_{3,3}$ has 9 edges and 6 vertices and is a bipartite graph. For it to be planar, it should verify lemma 2.2.5, so one should observe

$$9 \leq 2 \cdot 6 - 4 = 8$$

which is not the case. Therefore we may conclude that $K_{3,3}$ is nonplanar.



□

The desired result that $K_{3,3}$ and K_5 are nonplanar is enough to proof the 'only if' part of Kuratowski's theorem. This direction states that any graph containing a subdivision of $K_{3,3}$ or K_5 is nonplanar. This does intuitively make sense, as any graph with $K_{3,3}$ or K_5 in it, must have some nonplanar part and thus can not be drawn in the plane in it's entirety.

Theorem 2.2.7 (Kuratowski's theorem ($\Longrightarrow$)). Any graph containing a subdivision $K_{3,3}$ or K_5 as a subgraph is nonplanar.

Proof.

We prove the statement by contraposition, namely that a graph containing a Kuratowski subgraph is nonplanar. To see this, recall that both $K_{3,3}$ and K_5 are nonplanar by lemmas 2.2.6 and 2.2.3. Adding edges to a nonplanar graph cannot make it planar, so if a graph has a nonplanar subgraph, the graph itself is nonplanar. Similiarly, adding vertices on edges of a nonplanar graph will not make it planar, so all subdivisions of a nonplanar graph are nonplanar. Combining the results yields that a graph containing a subgraph being a subdivision of $K_{3,3}$ or K_5, then it is nonplanar. In other words, this states that a graph containing a Kuratowski subgraph is nonplanar, which is what we wanted.

□

We see that any graph containing a subdivision of $K_{3,3}$ or K_5 can not be planar. However, Kuratowski's theorem works in both directions. The 'if' direction implies that containing $K_{3,3}$ or K_5 are the only conditions needed to characterize planarity of graphs. There do not exist nonplanar graphs that do not contain a subdivision of $K_{3,3}$ or K_5. In section 3, we indeed proof that a graph either is planar, or contains a subdivision of $K_{3,3}$ or K_5.

3 Kuratowski's Theorem

In the last section we proved one part of Kuratowski's Theorem. The main focus of this section is to finish the proof of the entire theorem, so we concentrate on the "if" part during this section. First, we will recall the definition of a *Kuratowski subgraph*, along with the statement of the theorem.

Definition 3.0.1. [9] A graph contains a *Kuratowski subgraph* if it contains a subgraph which is a subdivision of K_5 or $K_{3,3}$.

Theorem 3.0.2 (Kuratowski's Theorem)**.** [9], [12] A graph G is planar if and only if it does not contain a Kuratowski subgraph.

Although all the definitions to understand this theorem have already been introduced, some key notions for the proof were left out. These include the definitions of *k-connectedness*, *edge contraction*, *inversion* and *S-lobes*. All these notions will be introduced in the following subsections, when they are needed. Throughout the first subsection, we shall familiarise ourselves with k-connectedness and edge contraction. Next, we shall link these notions with the concept of planar graphs. Finally, using inversion and S-lobes we will combine the notions of Kuratowski subgraphs and planar graphs, to get to the end of the theorem.

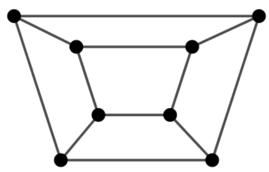
3.1 Some properties of 3-connected graphs

As many proofs in graph theory, we opt for a proof by induction. In order to do so, we need to prove that we may impose certain conditions on our graphs, which remain true even when reducing the graph in some way. In this section, we will prove that 3-connectedness is conserved through edge contraction. Since edge contraction reduces the number of vertices, this would constitute a good start for our induction step.

Two definitions that go hand in hand, and which we introduce in a single definition, is the notion of k-connectedness and the notion of the connectivity of a graph. Both notions give a quantitative value of the number of vertices it is possible to completely remove, including the edges adjacent to them, without making the graph disconnected.

Definition 3.1.1. [9] A graph is said to be *k-connected* if the removal of any $k-1$ vertices, along with the edges adjacent to these vertices, result in a connected graph. One defines the *connectivity* of a graph to be the largest k such that the graph is k-connected.

Example. The following graph is 3-connected. Indeed, no matter how hard one tries, removing 2 vertices from the graph cannot force the resulting graph to be disconnected. This means that the graph is also 2-, 1- and 0-connected. Since the graph is not 4-connected, we conclude its connectivity is 3.



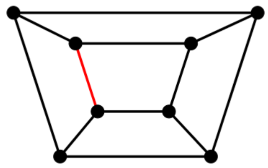
Along with the definition of k-connectedness we also define

Definition 3.1.2. [15] A *separating set* of a graph G is a set $S \subseteq V(G)$ such that $G-S$ has more than one component.

We will prove that the property of a graph being 3-connected is conserved under a certain transformation which reduces the number of vertices, which will allow us to use it for our proof by induction. The transformation we introduce is an *edge contraction*.

Definition 3.1.3. [9] An *edge contraction* is the process of merging the two endpoints of such an edge. If one performs such an action of a graph G with the edge e, we denote the resulting graph by G/e.

Example. In the following graph, the red edge has been contracted into the red vertex. Observe that the neighbours of the resulting vertex are exactly the neighbours of the endpoints of the initial edge.



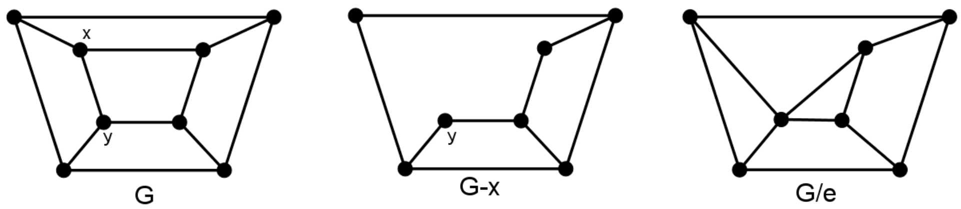
Definition 3.1.1 and 3.1.3 seem rather unrelated. The following proofs will relate them in various ways, which will prove themselves useful throughout the rest of the paper. To start of, we will prove that a 3-connected graph has an edge that can be contracted such that the resulting graph is 2-connected.

Lemma 3.1.4. [9] Let G be a 3-connected graph with $v(G) \geq 5$ and e an edge of G. Then G/e is 2-connected.

Proof.

Start by observing that $G - \{x\}$ has at least connectivity 2. Indeed, since G is 3-connected, removing any 2 vertices from G will conserve connectivity, so removing x and some other vertex will conserve connectivity. Thus, removing any vertex from $G - \{x\}$ conserves connectivity, which proves that $G - \{x\}$ is at least 2-connected.

Let e be the edge connecting x and y and notice that $G - \{x\} \subset G/e$. Indeed, if we assume the contraction of edge e contracts x and y into the vertex y, then all neighbours of y in G are also neighbours of y in G/e. This implies that G/e has connectivity at least 2, since $G - \{x\}$ has connectivity at least 2.



□

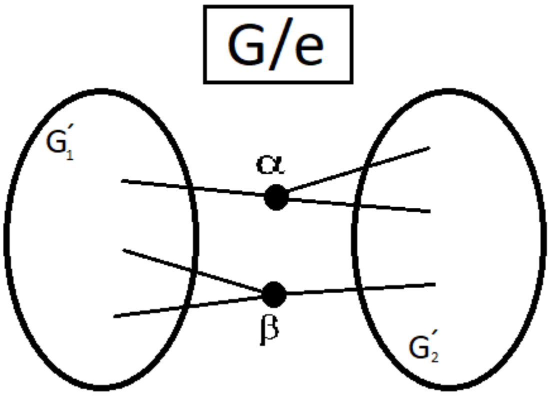
This simple result will build up stronger results. Indeed, this lemma proves that one can contract every edge and reduce the connectivity by at most 1. The result of the next subsection aims to prove is that if we pick the edge correctly, we may preserve the 3-connectedness. This might not be true for every edge, but our goal is to show that at least one such edge exists.

Since we only need one edge that preserves 3-connectedness, we can proceed by contradiction. We will suppose that performing an edge contraction on any edge results in a graph with connectivity exactly 2. In that case the resulting graph would have a separating set of size 2, which has some interesting properties leading to a contradiction. To get to the desired contradiction, we first introduce the next lemma.

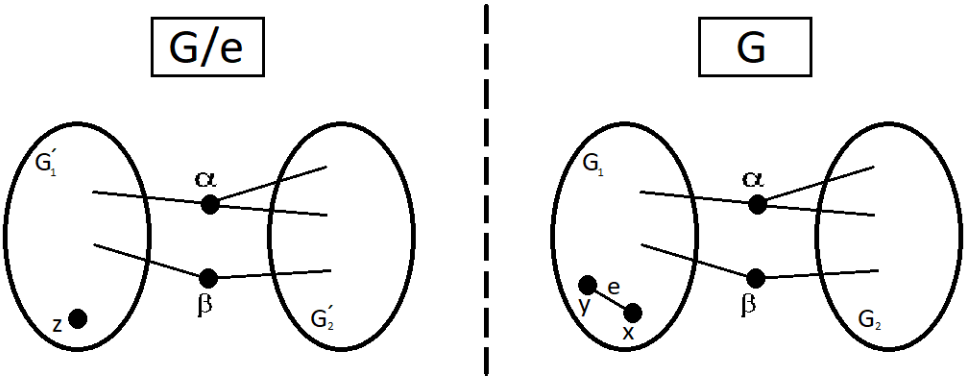
Lemma 3.1.5. [12] Let G be a 3-connected graph with $v(G) \geq 5$ and e an edge with endpoints x and y, such that G/e is at most 2-connected. Then G has a separating set of the form $\{x, y, w\}$ for some vertex w distinct from x and y.

Proof.

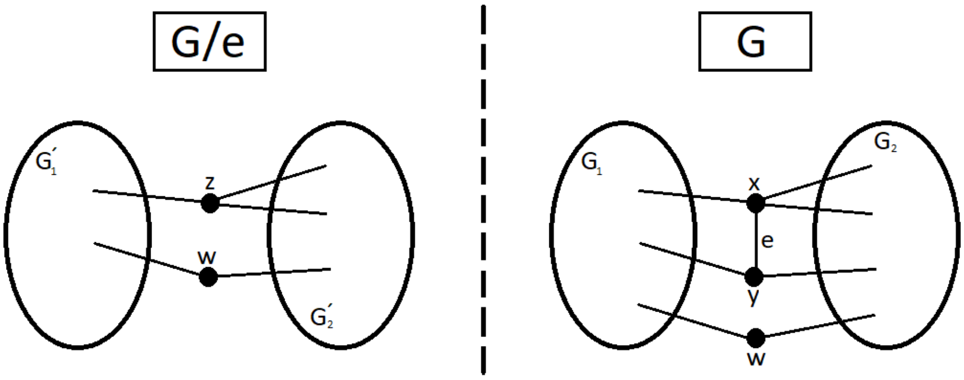
Firstly, we know that G/e has connectivity 2 by lemma 3.1.4. It thus has a separating set $\Lambda = \{\alpha, \beta\}$ of 2 vertices, which separates the graph into two disjoint sets G'_1 and G'_2.



Let z be the new vertex obtained after contracting e.
If $z \notin \Lambda$, then we may assume that z is contained in G'_1. Since G'_1 and G'_2 are disconnected, we know that all the neighbours of z are contained in G'_1. Let G_1 be the subgraph corresponding to G'_1 in G, such that G_1 contains both x and y. Then all neighbours of x and of y are in G_1, otherwise z would have a neighbour outside G'_1. Define G_2 similarly to G_1, such that nor x nor y are contained in G_2. Then G_1 and G_2 are disjoint, meaning that G has a separating set of 2 vertices. This is a contradiction since G is assumed to be 3-connected.



We may thus conclude that $z \in \Lambda$. Let w be the other vertex in Λ. Then $\{x, y, w\}$ is a separating set of G, as wanted.



$\square$

This result is applicable to proof that there exist an edge such that contracting it results in a 3-connected graph. Indeed, if every edge contraction results in an exactly 2-connected graph, then there exists a separating set of the form exposed in lemma 3.1.5, which will help us get the wanted contradiction.

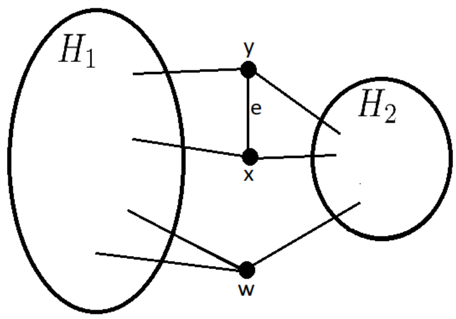
Another tool we use is the concept of Fermat's Infinite Descent. This method is used when trying to prove a certain property on a discrete set bounded from below. Usually, one supposes a solution satisfying the wanted property exists, and one takes the set of all solutions. Since the set of solutions is a subset of a discrete bounded by below set, it is itself a bounded discrete set, so it possesses a minimal element. One then picks this minimal element, and prove that there exists a solution even smaller. This would contradict the minimality of the solution, and thus contradicting the existence of solutions.

The concept of Fermat's Infinite Descent can also be applied to properties on discrete bounded by above sets, by assuming one can pick a maximal element and finding an even larger element. This is the variant which we will use in the following proof.

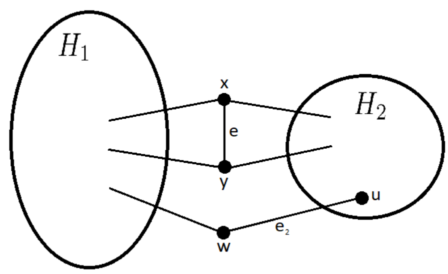
Lemma 3.1.6. [12] Let G be a 3-connected graph with $V(G) \geq 5$. Then there exists an edge e in G such that G/e is also 3-connected.

Proof.

We know that G/e is 2-connected for every edge e by Lemma 3.1.4. We will proceed by contradiction and suppose that G/e has **exactly** connectivity 2 for every edge e. Let x and y be the endpoints of some edge e. By lemma 3.1.5, there exists a vertex w distinct from x and y such that $\{x, y, w\}$ is a separating set of G. Pick e and w such that the largest component H_1 of $G - \{x, y, w\}$ has maximal size. Let H_2 be the other component of $G - \{x, y, w\}$.



Let u be a vertex in H_2 adjacent to w and let e_2 be the edge between w and u. We know such a vertex u exists, because the contrary would imply that $\{x, y\}$ is a separating set of G, which is absurd. Since any edge contraction yields a graph with connectivity 2. We know that G/e_2 has connectivity 2 (by assumption), lemma 3.1.5 states there exists a vertex v such that $\{w, u, v\}$ is a separating set of G.



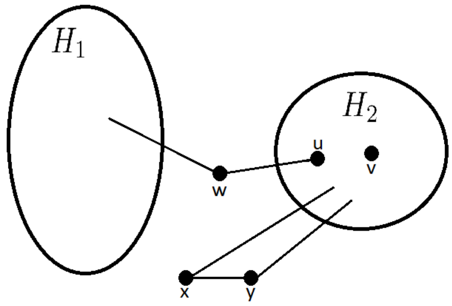
We now prove that

$$G(V(H_1) \cup \{x, y\}) - v$$

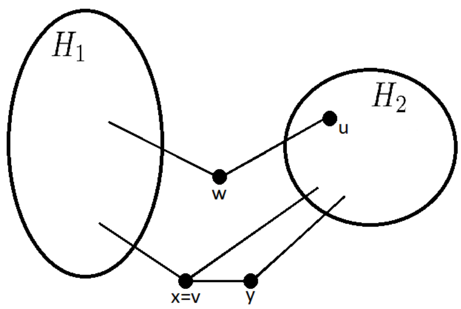
is connected.

To prove that, we shall consider multiple cases.

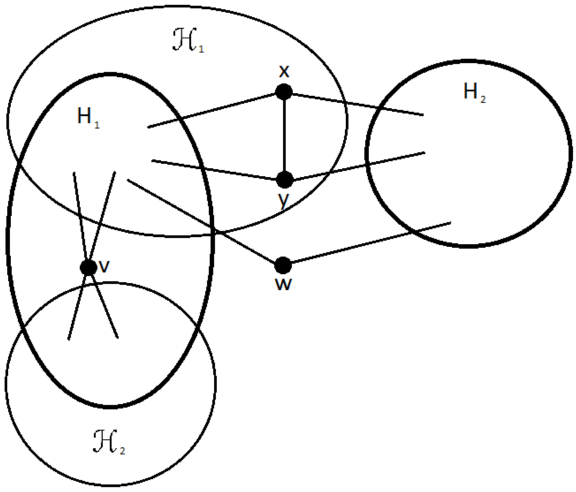
- If $v \in V(H_2)$, then
$$G(V(H_1) \cup \{x,y\}) - v = G(V(H_1) \cup \{x,y\})$$
If this set is not connected, then nor x nor y would have neighbours in H_1, and then $\{w\}$ would be a separating set of G, forcing G to have connectivity 1. This is of course a contradiction, so the given set must be connected.



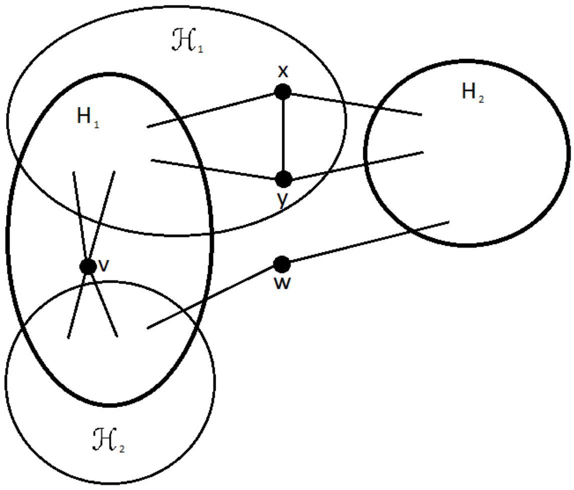
- If $v = x$, then:
$$G(V(H_1) \cup \{x,y\}) - v = G(V(H_1) \cup \{y\})$$
If this set were not connected, then y would not be connected to H_1, then $\{x = v, w\}$ is a separating set, contradicting the 3-connectedness of G. If $v = y$ the proof is similar.



- If $v \in V(H_1)$, suppose $G(V(H_1) \cup \{x,y\}) - v$ is not connected. We know that H_1 is connected to H_2 only via x, y and w, since $\{x,y,w\}$ separates the graph into these 2 components. We also know $G(V(H_1) \cup \{x,y\})$ is connected, otherwise one could remove either x or y from the separating set and thus obtain that G only has connectivity 2. So $\{v\}$ is a separating set of $G(V(H_1) \cup \{x,y\})$, and separates this graph into at least 2 disconnected components, $\mathscr{H}_1$ and $\mathscr{H}_2$. Not both components can be neighbours of w, since that would contradict their disconnectedness. Suppose that $\mathscr{H}_1$ contains an elements neighbour to w, but $\mathscr{H}_2$ does not.
  - If x and y are in $\mathscr{H}_1$, then $\{v\}$ would separate G into 2 disconnected components. Indeed, $\mathscr{H}_2$ is connected to $\mathscr{H}_1$ only via v and is not connected to x, y or w, so it cannot be connected to H_2. So $\mathscr{H}_1$ is a disconnected component of $G - \{v\}$, which contradicts the 3-connectedness of G.



– If x and y are not in $\mathscr{H}_1$, they must be in $\mathscr{H}_2$. Now this means that $\mathscr{H}_1$ is connected to $\mathscr{H}_2$ only via v, and to H_2 only via w. This means that $\mathscr{H}_1$ is a disconnected component of $G - \{v, w\}$, contradicting the 3-connectedness of G.



In every case we conclude that the subgraph

$$G(V(H_1) \cup \{x, y\}) - v$$

must be connected. This will contradict the maximality of the component H_1. Indeed, since nor w nor u are in $G(V(H_1) \cup \{x, y\})$ by definition, we know that $G(V(H_1) \cup \{x, y\})$ must be contained in some connected component H' of $G - \{w, u, v\}$. But

$$v(H_1) < v(G(V(H_1) \cup \{x, y\})) \leq v(H')$$

So a strictly larger component H' must exist, indeed contradicting the maximality of H_1. Since a maximal component H_1 cannot be picked, we conclude that G/e may not have connectivity 2 for every edge e. So G/e must have connectivity 3 for at least one edge e, as wanted.

□

To recapitulate, this lemma states that in a 3-connected graph, we can contract at least one edge without changing its 3-connectedness. This finally allows us to attack the proof of the 'if'-part of Kuratowski's theorem for 3-connected graphs.

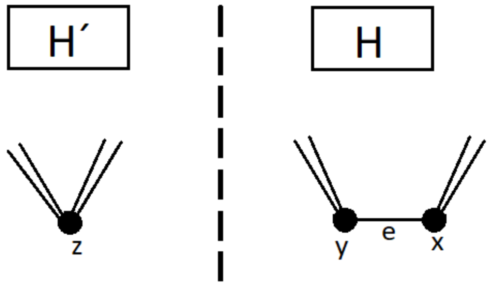
3.2 It must be planar

Recall that we would like to prove the result by induction. This is motivated by the fact that for any 3-connected graph, we can find an edge e such that G/e is 3-connected as well. Since we want to link 3-connectedness, Kuratowski subgraphs and planarity, we will prove that edge contraction conserves not containing Kuratowski subgraphs first.

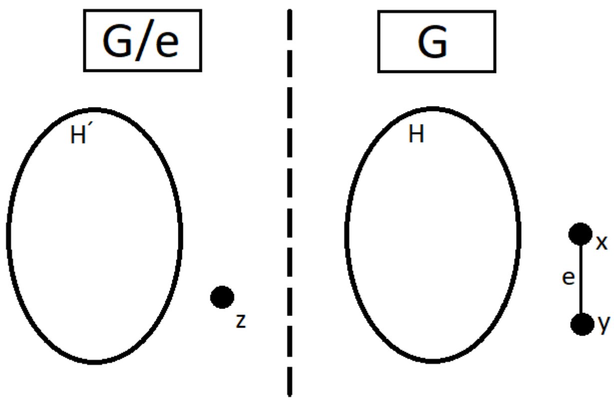
Lemma 3.2.1. [15] If a graph G has no Kuratowski subgraph, then neither does G/e for all edges e of G.

Proof.

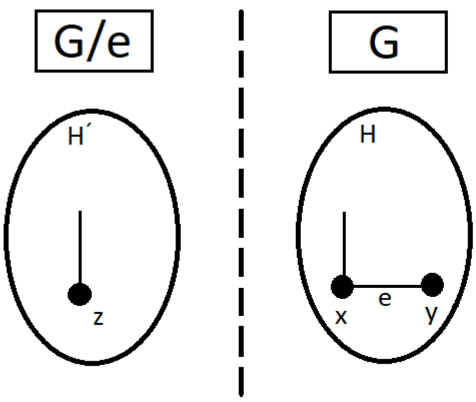
Suppose G/e were to contain a Kuratowski subgraph and take the subgraph H' a Kuratowski subgraph of G/e. Also denote by H the equivalent of H' in the graph G. Build H such a way that x and y do not share any neighbours. This means that the edges adjacent to x other than y in H and the edges adjacent to y other than x in H sum up to exactly the edges adjacent to z in H'.



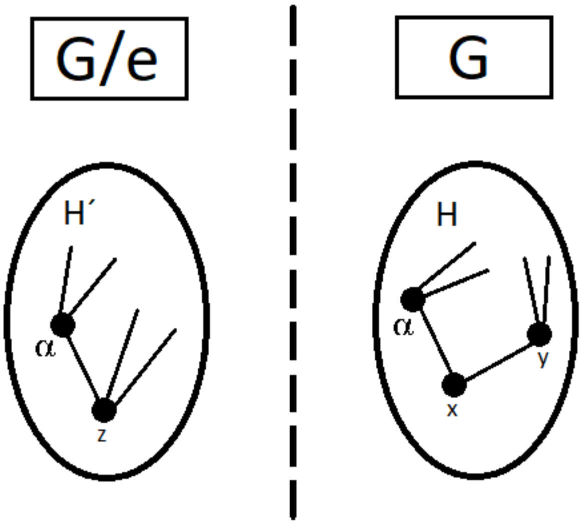
- If z is not a vertex of H', then H is also a subgraph of G and G would contain a Kuratowski subgraph. This is absurd.



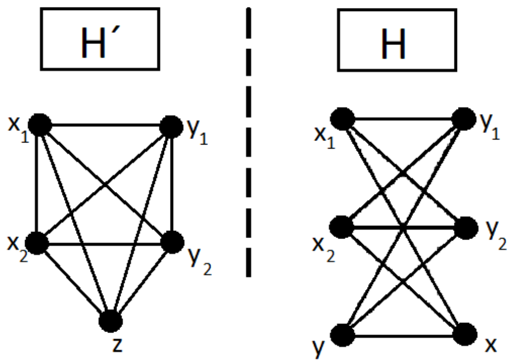
- If z is a vertex of H' such that the degree of z is 1, then we replace z by x in H . Then H is a subdivision of H', being also a Kuratowski subgraph, so G would contain a Kuratowski subgraph. This also leads to a contradiction.



- If z is a vertex of H' with degree more than or equal to 2 such that at most one edge adjacent to z in H' is adjacent to x in H, denote by α the other neighbour of x in H. Replacing z by y and the path from y to x to α by a path from y to α will result in a Kuratowski subgraph contained in G, which is again contradictory.



- If z is a vertex of H' with degree more than or equal to 2 such that at most one edge adjacent to z in H' is adjacent to y in H, then we are in an analogous case to the previous one, so it also leads to a contradiction.
- Finally, in the last case, z is a vertex with degree larger than or equal to 2 such that at least 2 edges adjacent to z in H are also adjacent to x in G, and at least 2 edges adjacent to y in H are also adjacent to z in H'. This means that z has at least 4 neighbours in H'. Since every vertex has at most 3 neighbours in a subdivision of $K_{3,3}$, we know that H' is a subdivision of K_5. Denote by x_1, x_2, y_1 and y_2 the four other vertices of the K_5 graph. Suppose x_1 and x_2 are connected by a path not containing y in H, and that y_1 and $y_@$ are connected by a path not containing x in H. Since z, x_1, x_2, y_1, y_2 form a subdivision of K_5, we know the pairs (x_1, x_2), (x_1, y_1), (x_2, y_2) and (y_1, y_2) to be connected by separate paths in H'. Since these paths do not contain z, they are the same in H. In addition to this, x is connected to x_1, x_2 and y, and y is connected to y_1, y_2 and x. This means that the 6 vertices x, y, x_1, x_2, y_1, y_2 form a subdivision of $K_{3,3}$ in H, so G again contains a Kuratowski subgraph. This case thus also leads to a contradiction.



We thus conclude G/e does not contain a Kuratowski subgraph.

□

We want to prove that certain graphs are planar. We shall prove that every 3-connected graph, not containing a Kuratowski subgraph, must be planar. This proof will be presented by induction, which is

highly motivated by the final lemma of the previous subsection, stating that there exists some edge such that contracting the edge conserves the 3-connectivity.

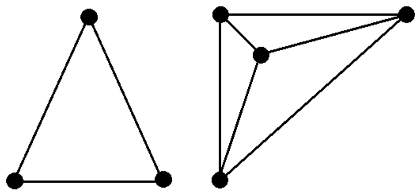
Theorem 3.2.2 (Tutte's Theorem)**.** [12] Let G be a finite 3-connected graph containing no Kuratowski subgraph. Then G is planar.

Proof.

We shall consider a proof by induction on the number of vertices on the graph G.

Base Cases.
We shall start the induction step at $v(G) \geq 5$, hence we should consider $v(G) < 5$ as the base cases. Note that graph G is 3-connected. The only 3-connected graph with $v(G) < 5$ must have $v(G) = 3$ and $v(G) = 4$. After all, a graph with less then 3 vertices is deleted completely by removing 2 vertices. The only 2 graphs with $v(G) = 3$ and $v(G) = 4$ that are 3-connected are K_3 and K_4.

- For $v(G) = 3$: For every removed vertex, the 2 remaining vertices must be connected. In other words, every pair of vertices must be connected, so the graph must be K_3, which is planar as seen in the following figure.
- For $v(G) = 4$: For every 2 removed vertices, the remaining 2 vertices must be connected. Since this must hold for any 2 vertices, we conclude that the only graph of 4 vertices for which this holds is the complete graph K_4, which is planar as seen in the figure.

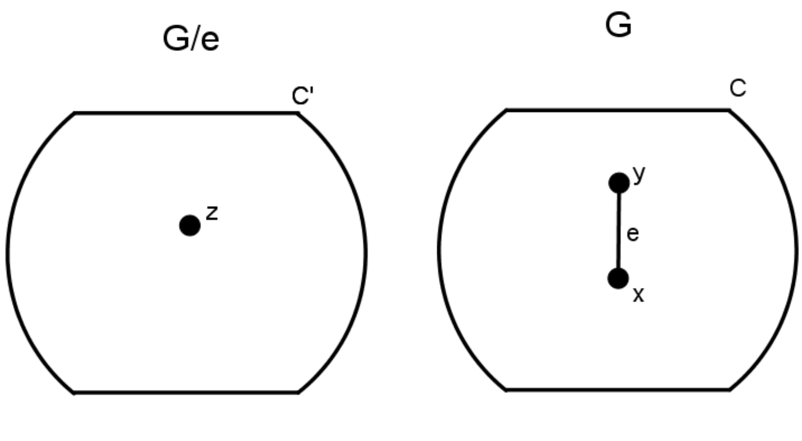


$K_{3,3}$ and K_4 respectively

Induction Step.
Suppose G is a 3-connected graph with $v(G) \geq 5$. Then there must exist an edge e of G such that G/e remains 3-connected by Lemma 3.1.6. Let x and y be the endpoints of this edge, and let z denote the "new" vertex created by contracting e.

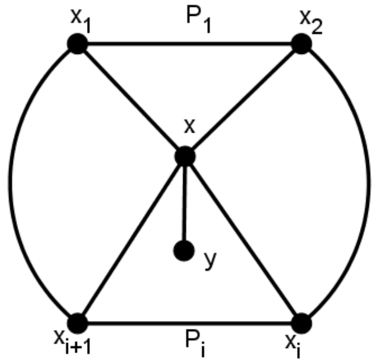
By lemma 3.2.1, G/e does not contain a Kuratowski subgraph, and since it is 3-connected, it satisfies the conditions of the lemma. Since it has less vertices than G, it must be planar by the induction hypothesis.

We embed graph G/e and identify C', the boundary of the face containing z.



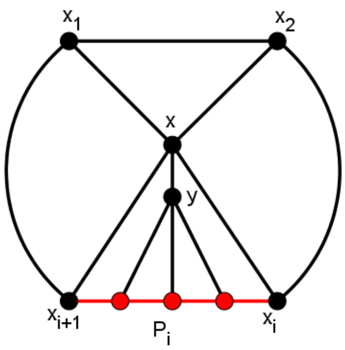
The 2 vertices x and y are contained in boundary C. We denote $x_1, x_2 \ldots x_k$ as the neighbours of x different from y in that order. Since G is a 3-connected graph, x has at least 3 neighbours (including y),

so $k \geq 2$. We denote P_i by the segment on C between the two neighbours x_i and x_{i+1} $(i = 1, \cdots k)$.

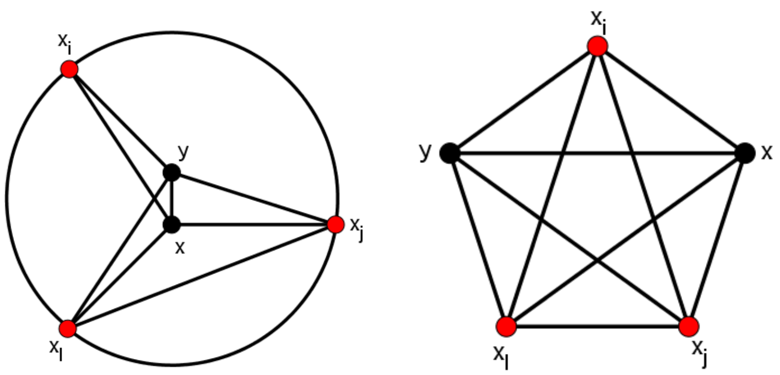


We distinguish between a few cases:

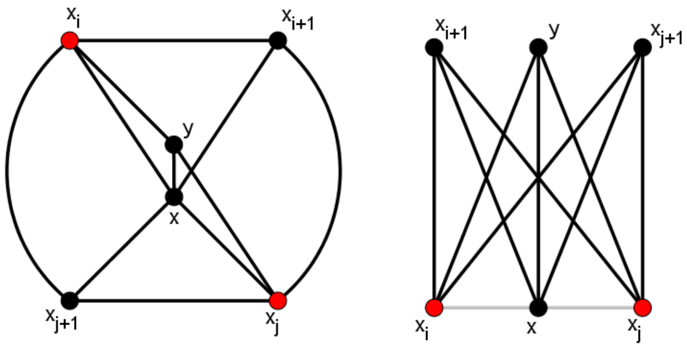
- If all neighbours of y (except for x) are on the same segment P_i, we can find a planar embedding for G. We embed G/e and take vertex z to be x. Next put vertex y in the face bounded by xx_i, xx_{i+1} and segment P_i. Next we can connect y to it's neighbours on segment P_i without having the edges intersect. We conclude that G must be planar.



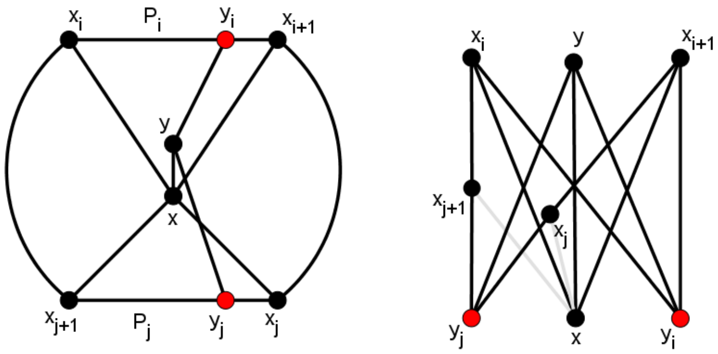
- If 3 or more neighbours of y are connected to neighbours of x, say x_i,x_j and x_l, we find that these vertices together with x and y create a subdivision of K_5. G contains a Kuratowski subgraph, which is a violation of the initial conditions, so this may not be the case.



- Next, y can also be connected to exactly 2 non-adjacent neighbours x_i and x_j. Observe that x_{i+1} and x_{j+1} are between x_i and x_j, on opposite sides, since they are not adjacent. In this case, these four vertices along with x and y, form a subdivision of $K_{3,3}$. Again, G contains a Kuratowski subgraph and the conditions of the theorem are violated, so this case is impossible.



- Finally, if y has 2 or more neighbours that are on different segments P_i and P_j ($i \neq j$), say y_i and y_j, then the six vertices x, y, x_i, y_i, x_j and y_j form a subdivision of $K_{3,3}$, so this would also violate the conditions of the theorem, and thus this case is also impossible.



We have thus observed that if G satisfies the conditions of the theorem, the neighbours of y must be placed as stated in the first case, in which case we can conclude that G is planar. All other positions of the neighbours of y result in the conditions not being met.

□

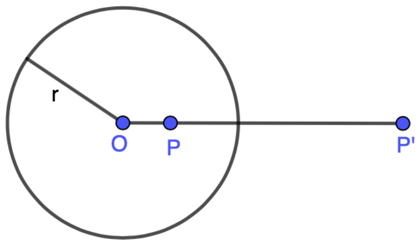
With this result, we have done the first half of the proof for the 'if' part of Kuratowski's theorem. The result states that for any 3-connected graph, the graph is planar if it does not contain a Kuratowksi subgraph. Next, we generalize the statement to all graphs. For this, we need to introduce some new notions.

3.3 Nonplanarity implies 3-connectedness

In order to avoid too large case testing, we want to show that for any vertex of a planar graph, one can embed the graph in a planar way such that the vertex is adjacent to the unbounded face.

In order to do so, we introduce a transformation of the plane which sends objects that are close to a point far away and vice versa. We can transform the boundary of any face to be the unbounded face by applying this inversion to a point within that face.

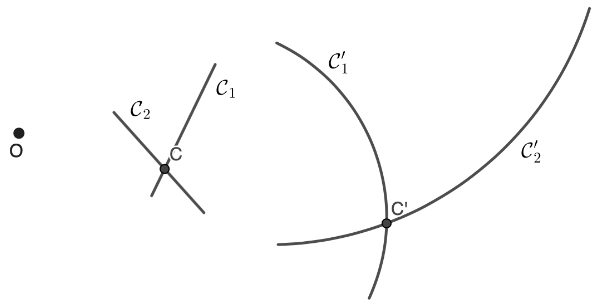
Definition 3.3.1. *Inversion* is the process of transforming points P to a corresponding set of points P' known as their inverse points. Two points P and P' are said to be inverses with respect to an *inversion center* O and *inversion radius* k if O, P and P' are aligned, with P and P' on the same side of O, and placed such that $\overline{OP} \cdot \overline{OP'} = r^2$.



In order to be able to use this transformation, it would have to preserve planarity of a graph. To simplify the proof, we show that two non-intersecting curves will not intersect after inversion. Extending this on a graph will show that a planar graph remains planar after inversion.

Proposition 3.3.2. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two curves with no intersections in any points other than their endpoints. Let O be a point not on these curves. Then the images of $\mathscr{C}_1$ and $\mathscr{C}_2$ under an inversion of center O do not intersect in any points other than their endpoints.

Proof. Suppose that $\mathscr{C}_1'$ and $\mathscr{C}_2'$ intersect at some point C' different from their endpoints. The point C, the image of C', lies on the images of $\mathscr{C}_1'$ and of $\mathscr{C}_2'$. Since applying the inversion twice results in identity, these images are $\mathscr{C}_1$ and $\mathscr{C}_2$. This implies that $\mathscr{C}_1$ and $\mathscr{C}_2$ intersect in C, a point different from their endpoints, which is a contradiction. □



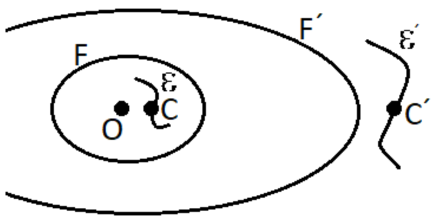
Lemma 3.3.3. Let F be the set of edges forming the boundary of a face in a planar embedding of G. Then there exists a planar embedding of G where F forms the boundary of the unbounded face.

Proof.

Pick any point O strictly inside the face F, and apply an inversion of center O to the entire graph. By Proposition 3.3.2, any two edges in the inverted graph will intersect at most at their endpoints, so the inverted graph remains planar.
Suppose the image of F, F', does not form the boundary of the unbounded face of the inverted graph. Then there must exist some part of an edge $\mathscr{E}'$ outside of F. Pick a point C' on $\mathscr{E}'$. Denote by C the inverse of C', and by $\mathscr{E}$ the inverse of $\mathscr{E}'$. Also, denote by F_1 and F_2 one intersection of CC' with F and F' respectively. Since the edge is outside of the face F', we know that

$$\overline{OF'} < \overline{OC'} \implies \frac{r^2}{\overline{OF'}} > \frac{r^2}{\overline{OC'}} \implies \overline{OF} > \overline{OC}$$

So the point C is strictly inside the face F before inversion, which proves there is an edge inside of the face. This is a contradiction, since this would imply F is not the boundary of a face.



□

This finishes the inversion needed to proof the existence of an embedding such that any face is the unbounded face.

To continue, recall that any 3-connected graph containing no Kuratowski subgraph is planar by lemma 3.2.2. Next in the proof, we want to show that a minimal nonplanar graph not containing a Kuratowski subgraph is 3-connected, forcing the graph to also be planar. This would be a contradiction, completing the 'only if' part of Kuratowski's theorem.

Since we are aiming for a contradiction, we might impose additional restrictions. To make the proof simpler, we consider a nonplanar graph such that the removal of any edge forces it to be planar. In section 3.4 we prove that if the result is true for such graphs, it must be true for every graph.

Definition 3.3.4. A subgraph H of G is called *proper* if $v(H) < v(G)$ or $e(H) < e(G)$.

Definition 3.3.5. [12] A *minimal nonplanar graph* is a nonplanar graph such that every proper subgraph is planar.

To start, we prove that any minimal nonplanar graph is 2-connected.

Lemma 3.3.6. [12] If G is a minimal nonplanar graph, then G is 2-connected.

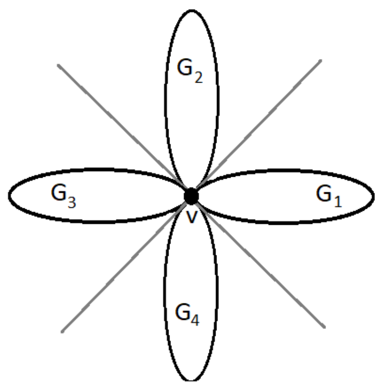
Proof.

We prove the above statement by contradiction. Suppose that G is not 2-connected, so that G has connectivity at most 1. Then, we distinguish two cases:

- If G has connectivity 0, then G is disconnected. G being a minimal nonplanar graph, the disconnected components of G are planar. But if G is composed of a number of planar components, it must itself be planar, contradicting the assumption that G is a minimal nonplanar graph.
- If G has connectivity 1, it has a cut-vertex (separating set of cardinality 1) v. Then, we denote the disconnected components of $G - \{v\}$ by $H_1, H_2, \ldots, H_k$. Next, define

$$G_i = G(V(H_i) \cup \{v\})$$

Notice that $k \geq 2$, elsewise v would not be a cut-vertex. Since G is a minimal nonplanar, G_i is a proper subgraphs for all i, so it is planar. By Lemma 3.3.3, we can find a planar embedding for each G_i with v on the unbounded face. We construct a planar embedding of G by combining the planar embeddings of G_i at v in such a way that each embedding fits in an angle smaller than $\frac{2\pi}{k}$ radians at v. The constructed graph contradicts the assumption that G is a nonplanar graph, so it certainly cannot be a minimal nonplanar graph.



□

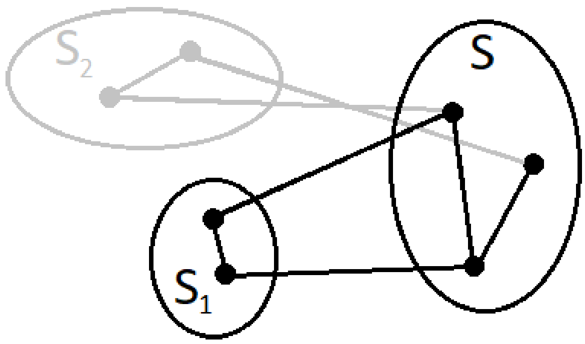
We would like to extend this proof such that a minimal nonplanar graph is also 3-connected. We will suppose it is not and find some nonplanar proper subgraph of the graph. That would contradict the minimality of the nonplanar graph, and thus yield the wanted result.

To find such subgraph, we introduce S-lobes. Informally, an S-lobe is just a subgraph of G, containing the set of vertices S and the vertices of some disconnected component of $G-S$.

Definition 3.3.7. [15] Let S be a set of vertices on a graph G. Then, an S-*lobe* of G is an induced subgraph of G whose vertex set consists of S and the vertices of a disconnected component of $G-S$.

This definition might seem rather arbitrary, so it is useful to provide a well-constructed example.

Example. Firstly, we pick some set of vertices S in G. Then we identify all the disconnected components of $G-S$, being S_1 and S_2 in this case. Then the graph contains two S-lobes, namely $G(S \cup V(S_1))$ and $G(S \cup V(S_2))$.

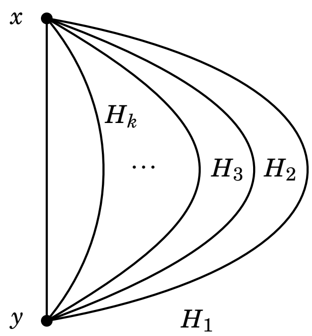


Lemma 3.3.8. [15] Let $S = \{x, y\}$ be a separating set of a nonplanar graph G and let H be a S-lobe of G. Then $H \cup \{xy\}$ is nonplanar.

Proof.

We prove the contrapositive statement which reads as follows: If $H \cup \{xy\}$ is planar for all S-lobes H of G, then G must be planar.

Let $G_1, G_2, \ldots, G_k$ be the S-lobes of G and define $H_i = G_i \cup \{xy\}$. Suppose that all graphs H_i are planar. By lemma 3.3.3, we can embed H_1 such that the edge xy is placed on the boundary of the unbounded face. Then, all graphs $H_1, H_2, H_3, \ldots, H_k$ can be embedded together such that $G(V(H_1) \cup \cdots \cup V(H_k)) \cup \{xy\}$ is planar.



□

If some nonplanar graph not containing a Kuratowski subgraph would exist, we could take the one with the fewest edges and reach some contradiction, which proves that such graphs do not exist. This contradiction arises if we manage to prove that the minimal nonplanar graph not containing a Kuratowksi subgraph is 3-connected. This is due to the fact that every 3-connected graph not containing a Kuratowski subgraph has been proven planar by lemma 3.2.2.

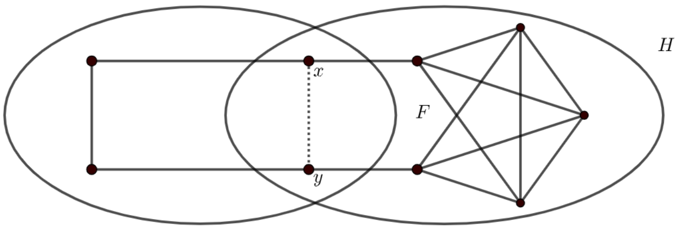
Lemma 3.3.9. [15] If G is a graph with fewest edges among all nonplanar graphs without a Kuratowski subgraph, then G is 3-connected.

Proof.

We prove the above statement by contradiction. Suppose G satisfies all the conditions but is not 3-connected.

Since G is a graph with the fewest edges among all nonplanar graphs without a Kuratowski graph, removing any edge from G makes it planar or produces a graph containing a Kuratowski subgraph. Observe that we cannot obtain a Kuratowski subgraph by deleting an edge from the graph G since G does not contain any such subgraph, so removing any edge from G produces a planar graph. This implies that G is a minimal nonplanar graph, so, by lemma 3.3.6, G is 2-connected.

Assume that G is not 3-connected, so that it has connectivity exactly 2. Then, there exists a separating set $S = \{x, y\}$ of G. By lemma 3.3.8, we know that $H = H' \cup \{xy\}$ is nonplanar for some H' being an S-lobe of G. Note that the graph H has fewer edges than the graph G because $G - H$ cannot be empty and both x and y have neighbours in $G - H$. Then, the minimality of G implies that H must contain a Kuratowski subgraph F. However, F need not be contained in G since the edge xy is not necessarily an edge of G. Nevertheless, note that since S is a separating set, both elements x and y have neighbors in every S-lobe. Hence, to construct a Kuratowski subgraph contained in G, we replace the edge xy in F by a path from x to y through another S-lobe of G.



We do thus observe a Kuratowski subgraph in the graph G, which contradicts the initial conditions. Hence, G does not contain a separating set with two vertices, contradicting the assumption. We may thus conclude that G is indeed 3-connected, as wanted.

□

3.4 One final step

We have proved in lemma 3.2.2 that a 3-connected graph not containing a Kuratowski subgraph is planar, and by lemma 3.3.9 that the nonplanar graph not containing a Kuratowski subgraph with the fewest edges is 3-connected. Combining these thus tells us that the nonplanar graph not containing a Kuratowski subgraph with the fewest edges must be planar, which is a contradiction.

Theorem 3.4.1 (Kuratowski's Theorem)**.** [9], [12], [15] A graph G is planar if and only if it does not contain a Kuratowski subgraph.

Proof.

Only if. By theorem 2.2.7 we know that the only if part of Kuratowski's theorem holds. Indeed, any graph containing a Kuratowski subgraph cannot be drawn in the plane without intersection.

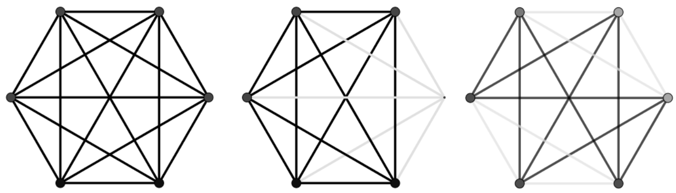
If. We prove the statement by contradiction. Suppose there exists a non-empty set of nonplanar graphs that do not contain a Kuratowski subgraph and take G to be the graph in this set with the fewest edges. Then lemma 3.3.9 proves that this graph G is 3-connected. Recall that G does not contain a Kuratowski subgraph, so lemma 3.2.2 implies that G is also planar. Since G is supposed nonplanar, this is a contradiction and there does not exist a non-empty set of nonplanar graphs that do not contain a Kuratowski graph. This proves that the given set is empty, and that being nonplanar and containing a Kuratowski subgraph are mutually excluding.

This means that if a graph is nonplanar, it must contain a Kuratowski subgraph, which is the contrapositive statement of the wanted result.

□

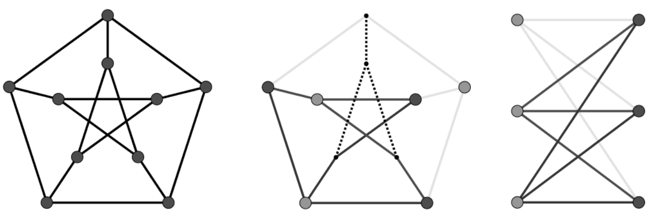
This result gives an entire characterization of planar graphs. Indeed, instead of drawing the graph out in multiple ways and checking whether it can be drawn without intersections, one only needs to test for subgraphs of subdivisions of $K_{3,3}$ or K_5.

The complete graph with 6 vertices, K_6, is nonplanar. This can we seen in 2 ways, since it contains both K_5 and $K_{3,3}$.



K_6, K_5 and $K_{3,3}$ respectively

The Peterson graph is nonplanar. This can be observed because of the subdivision of $K_{3,3}$ it contains.



The Peterson graph (Leftmost graph), $K_{3,3}$ (Middle and rightmost graph)

4 Generalisations

After proving Kuratowski's Theorem (3.4.1), one might wonder if it is possible to make a similar statement for different surfaces. It turns out this is possible. Although the results are similar, the exact graphs that characterize which structures cause graphs to have intersecting edges can differ a lot. Throughout this section we analyse these graphs on different surfaces.

4.1 Kuratowski on a Sphere

It might come as a surprise to the reader, but Kuratowski's theorem also holds on a sphere [1]. We of course do not go through the entire proof of the theorem again, so to show that it holds on a sphere we use the result proven earlier, namely that it holds in the plane.

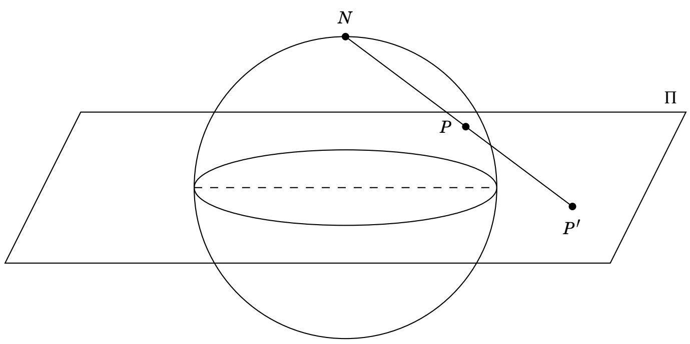
Notice that the proof exposed in section 3 does not hold for a sphere. Indeed, we use multiple times that any graph must have an unbounded face, as for instance in lemma 3.3.3. On a sphere we cannot define an unbounded face, at least not easily, which make the previous proof non-transferable to a sphere. In this subsection we will take a look at a simple and visual way of convincing ourselves that Kuratowski's result holds on a sphere.

Firstly, we will define the analogous definition of a *planar* graph on a sphere.

Definition 4.1.1. A *spherical graph* is a graph that can be embedded on a sphere without its edges intersecting.

Since our aim is to prove that spherical and planar are equivalent, we will introduce a transformation which maps a spherical graph to a planar graph.

Definition 4.1.2. [4] Let S^2 be a sphere, N a point on S^2 and Π a plane in three dimensions. The *stereographic projection* of the sphere S^2 from the projection center N onto the plane Π is a geometric transformation which maps every point $P \neq N \in S^2$ to P' the intersection of NP with Π.



One can study this transformation more closely to get a closed formula for it. We will consider $S^2 = \{(x,y,z)|x^2+y^2+z^2=1\}$ the unit sphere, $N=(0,0,1)$ the north pole of the sphere and $\Pi \equiv z=0$. We shall call this the **standard** stereographic projection.
Let (x,y,z) be a point on S^2 and let (X,Y) be the projected point on Π. We know that these two points are aligned with N, so

$$\frac{x-0}{X-0} = \frac{y-0}{Y-0} = \frac{z-1}{0-1}$$
$$\Longrightarrow (X,Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

Denote the stereographic projection of a point (x,y,z) by $\phi((x,y,z))$ such that

$$\phi\colon \{(x,y,z)|(x,y,z) \in \mathbb{R}^3 \setminus \{0,0,1\}, x^2+y^2+z^2=1\} \mapsto \mathbb{R}^2,$$
$$\phi((x,y,z)) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

We can also define the inverse of the stereographic projection. For any point (X,Y) on the plane $z=0$, we take the point (x,y,z) on the unit sphere such that N, (x,y,z) and $(X,Y,0)$ are aligned. This means that

$$\frac{x-0}{X-0} = \frac{y-0}{Y-0} = \frac{z-1}{0-1}$$

$$\implies x = X(1-z), y = Y(1-z)$$

Recall that $x^2+y^2+z^2=1$

$$\begin{aligned}
&\implies (X^2+Y^2)(1-z)^2+z^2=1\\
&\implies (X^2+Y^2)(1-z)=1+z\\
&\implies z(X^2+Y^2+1)=X^2+Y^2-1\\
&\implies (x,y,z)=\left(\frac{2X}{X^2+Y^2+1},\frac{2Y}{X^2+Y^2+1},\frac{X^2+Y^2-1}{X^2+Y^2+1}\right)
\end{aligned}$$

Denote this inverse function by

$$\psi:\mathbb{R}^2\mapsto\{(x,y,z)|(x,y,z)\in\mathbb{R}^3\setminus\{0,0,1\},x^2+y^2+z^2=1\},$$

$$\psi((X,Y))=\left(\frac{2X}{X^2+Y^2+1},\frac{2Y}{X^2+Y^2+1},\frac{X^2+Y^2+1}{X^2+Y^2+1}\right)$$

Using this we prove that the stereographic projection ϕ is a bijection. Before this, we will show that $\phi\circ\psi\equiv Id\equiv\psi\circ\phi$

$$\begin{aligned}
\psi(\phi((x,y,z))) &= \psi\left(\left(\frac{x}{1-z},\frac{y}{1-z}\right)\right)\\
&=\left(\frac{2\frac{x}{1-z}}{\frac{x^2}{(1-z)^2}+\frac{y^2}{(1-z)^2}+1},\frac{2\frac{y}{1-z}}{\frac{x^2}{(1-z)^2}+\frac{y^2}{(1-z)^2}+1},\frac{\frac{x^2}{(1-z)^2}+\frac{y^2}{(1-z)^2}-1}{\frac{x^2}{(1-z)^2}+\frac{y^2}{(1-z)^2}+1}\right)\\
&=\left(\frac{2x(1-z)}{1-z^2+(1-z)^2},\frac{2y(1-z)}{1-z^2+(1-z)^2},\frac{1-z^2-(1-z)^2}{1-z^2+(1-z)^2}\right)\\
&=(x,y,z)\\
\phi(\psi((X,Y))) &= \phi\left(\left(\frac{2X}{X^2+Y^2+1},\frac{2Y}{X^2+Y^2+1},\frac{X^2+Y^2-1}{X^2+Y^2+1}\right)\right)\\
&=\left(\frac{\frac{2X}{X^2+Y^2+1}}{1-\frac{X^2+Y^2-1}{X^2+Y^2+1}},\frac{\frac{2Y}{X^2+Y^2+1}}{1-\frac{X^2+Y^2-1}{X^2+Y^2+1}}\right)\\
&=\left(\frac{\frac{X}{X^2+Y^2+1}}{\frac{1}{X^2+Y^2+1}},\frac{\frac{Y}{X^2+Y^2+1}}{\frac{1}{X^2+Y^2+1}}\right)\\
&=(X,Y)
\end{aligned}$$

This implies that the stereographic projection ϕ is a bijection:

- ϕ is a surjection. This is true since for every point $\mathbf{x}$ there always exists a point $\mathbf{y}$ such that $\phi(\mathbf{y})=\mathbf{x}$. Specifically, we may pick $\mathbf{y}=\psi(\mathbf{x})$.
- ϕ is an injection. Suppose that $\phi(\mathbf{y})=\phi(\mathbf{z})$. Then $\psi(\phi(\mathbf{y}))=\psi(\phi(\mathbf{z}))$, which implies that $\mathbf{y}=\mathbf{z}$. This means that ϕ is indeed an injection.

Since the projection is both a surjection and an injection, we may conclude it is a bijection.

Since the projection is a bijection, we know that two non-intersecting curves on the sphere will not intersect after stereographic projection. If they were to do so, one could perform the inversion projection to observe that both curves passed through the same point, which contradicts their non-intersection.

In specific, we may conclude that a graph drawn on S^2 can be stereographically projected onto Π such that planarity is conserved. To ensure the projected graph is contained in the plane, we just have to ensure the projection point is strictly included in a face of the graph. Since the transformation is conserved, we might even observe that there is a bijection between spherical graphs and planar graphs.

This means that a graph is spherical if and only if it is planar. Combining this with Kuratowski's theorem from the earlier section, we conclude the following lemma.

Lemma 4.1.3. [15] A graph is spherical if and only if it does not contain a Kuratowski subgraph.

Note that one could also adapt the previous proof to the sphere by considering the unbounded face as the face containing the north pole of the sphere. If the north pole is contained on an edge, we simply shift the graph to make sure it does not. With a little work, the given proof of Kuratowski's theorem then applies to the sphere as well.

4.2 Other surfaces

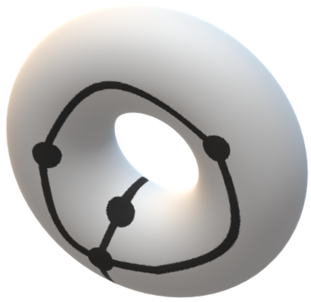
We have proven Kuratowski's Theorem (3.4.1) for the plane and a sphere (lemma 4.1.3), we might wonder whether or not the result holds for other surfaces. Unfortunately, this is not the case [10]. One might already be aware of that after observing that some parts of the proof rely on the graphs being drawn in the plane.

To gain an intuition for why the result is not true for any surface, we might look back at theorem 2.1.6, Euler's Characteristic Formula.

Theorem 2.1.6 (Euler's Characteristic Formula)**.** [3], [14] For any connected planar graph G, the following holds

$$v - e + f = 2$$

This result is proper to the plane. To convince yourself of its non-validity on other surfaces, we might look at this simple graph on a torus. It has 4 vertices, 5 edges, and 1 face. Then we observe that $v(G)-e(G)+f(G) = 0$, and not 2 as wanted.



Although the exact result does not hold on every other surfaces, the result might be extended to a more general result.

Theorem 4.2.1 (Generalised Euler Characteristic Formula)**.** [11] For a graph G drawn on a surface S, the following holds

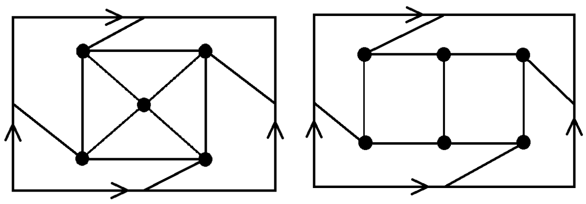
$$v(G) - e(G) + f(G) = \chi(S)$$

where $\chi(S)$ is the Euler Characteristic of the surface.

It is worth noting that if one draws the graph on a surface S containing a surface S', such that the graph could have been drawn on S', then the result holds with $\chi(S')$. Indeed, if one draws a graph on a torus without using the hole, it could have been drawn on a plane.

We do not provide the proof of the theorem, but the proof is very similar to the one exposed in an earlier section. Indeed, one can proceed by induction and reduce the graph to a base case, whilst keeping the value of $v(G)-e(G)+f(G)$ constant. In the base case it will be equal to $\chi(S)$, so the formula would have been true for the original graph as well.

Having introduced the generalised Euler Characteristic Formula, we can clearly see why the proofs of corollary 2.2.3 and lemma 2.2.6, namely that K_5 and $K_{3,3}$ are nonplanar, do not hold on different surfaces. For the torus, we might observe the following embeddings, proving that the result does definitely not hold [13].



K_5 and $K_{3,3}$ embedded on a torus

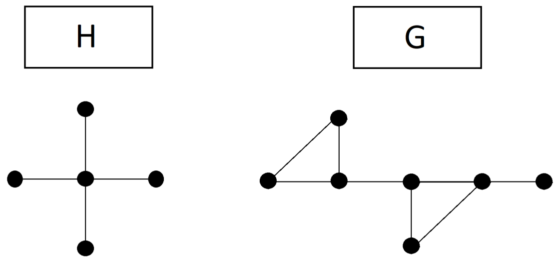
This also proves that we cannot have $\chi(T) = 2$, where T is the surface of a torus. Indeed, if one does the computations, one will observe that $\chi(T) = 0$.

Although Kuratowski's characterization does not hold for a most surfaces, an analogous result does hold.

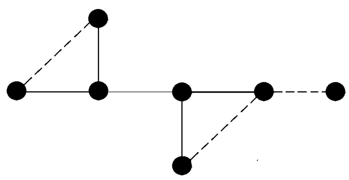
Theorem 4.2.2 (Robertson-Seymour Theorem)**.** [10] Every minor-closed family of graphs is defined by a finite set of forbidden minors.

Before being able analyze this result, we need to be sure every notion is well understood. Firstly, a graph H is called a minor of a graph G is H has a subdivision that is a subgraph of G. Secondly, a minor-closed family of graphs is a family of graphs all respecting a certain property, such that any minor of these graphs also respect this property. In other words, the property is conserved under taking minors, or the family is closed under taking minors. Finally, a set of forbidden minors defines a family if the family is composed of exactly all graphs not containing some specific minors, namely the forbidden minors.

To illustrate the concept of forbidden minors we might observe a short example.



In this example, H is a minor of the graph G. Indeed, we observe that a subdivision of H is a subgraph of G. This subdivision looks as follows



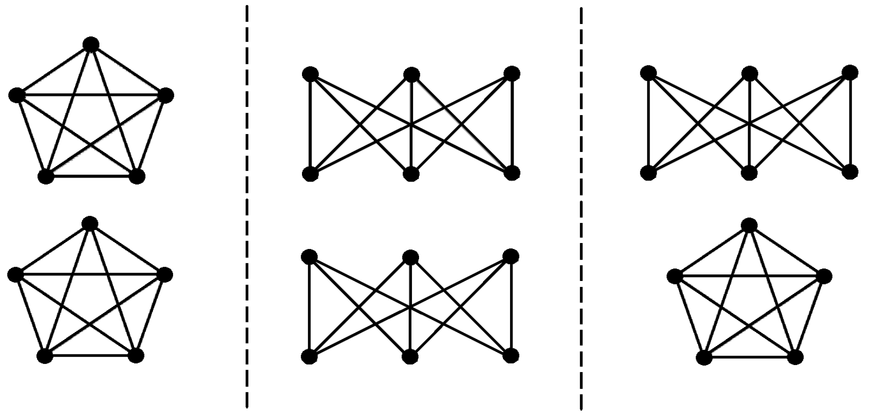
Now that we understand the statement of the proof, we might look at its implications.

First, we observe that the family of all graphs is clearly minor-closed. Theorem 4.2.2 then states that this family is defined by a finite set of forbidden minors. This is indeed true, taking the set of forbidden minors to be the empty set.

Next, one can observe the family of planar graphs. It is also minor-closed, since every minor of a planar graph remains a planar graph. The theorem thus states that the family is defined by a finite set of forbidden graphs, which Kuratowski's theorem proves to be the set $\{K_5, K_{3,3}\}$.

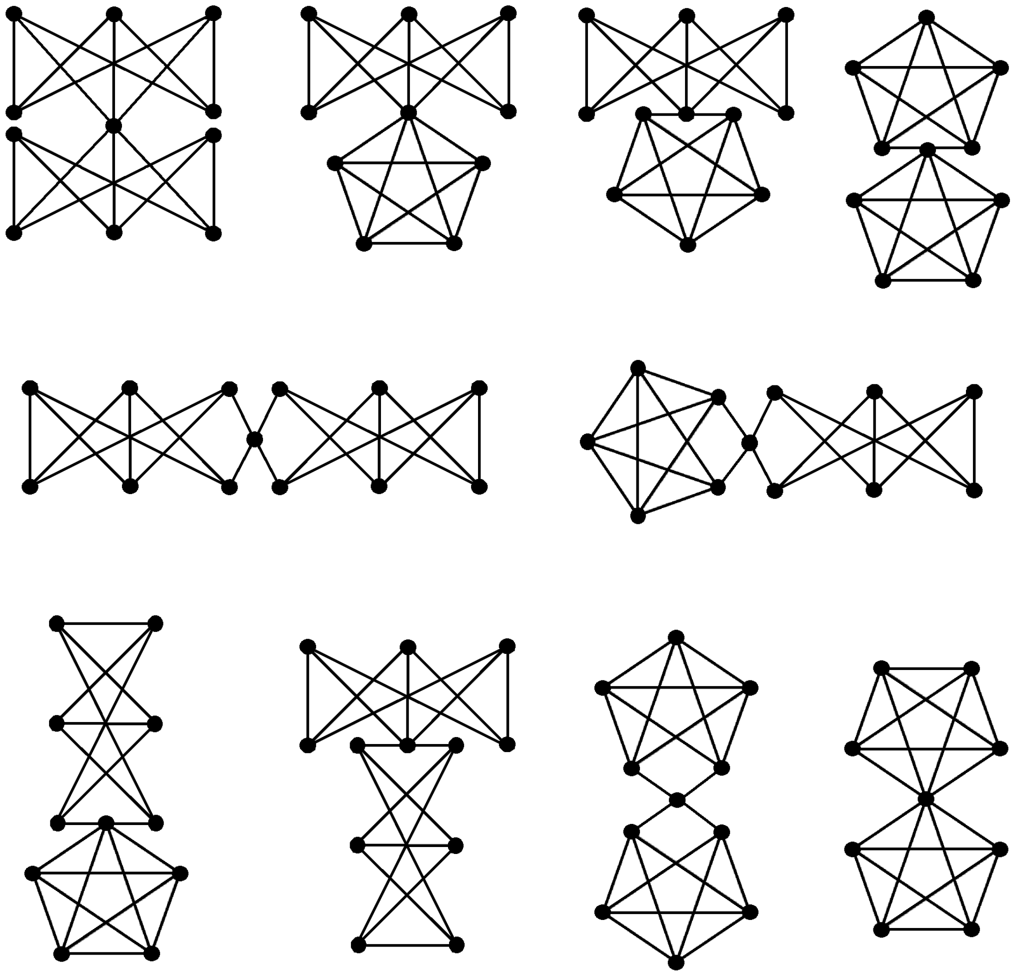
Finally, this theorem proves that there exists a finite set of forbidden graphs that defines the family of graphs that can be embedded on a torus without intersections. Although this set is proven finite, it is not small nor known. To this date, one only knows that the set is composed of at least 17535 different forbidden minors [10].

Although $K_{3,3}$ and K_5 are not forbidden minors for a torus, some very similar graphs to these form forbidden minors on the torus. Firstly, observe that even disconnected graphs might be forbidden minors. The 3 following disconnected graphs constitute a non-exhaustive list of disconnected forbidden minors, and thus non-embeddable without intersections on a torus.



Graphs composed of 2 components of K_5 or $K_{3,3}$

Of course there are a lot of other forbidden minors, of which we expose a few in the following graphs.



5 Conclusion

Throughout the article, we addressed the question whether a graph can be embedded without intersections on different surfaces. At first, we considered the case of the Euclidean plane. We have seen that graphs that can be drawn in the plane without intersections are called planar graphs. Kuratowski's theorem provides a complete characterization of such graphs.

We explored basic properties of planar graphs. In particular, we gave an elementary proof of Euler's characteristic formula for the plane. This remarkable result links the number of vertices, the number of edges and the number of faces by $v - e + f = 2$. We also derived upper bounds for the number of edges as a function of the number of vertices for planar graphs. Since $K_{3,3}$ and K_5 do not satisfy these conditions, we concluded that these graphs can never be embedded in the plane without intersections. From this, we were able to give a short proof of the 'only if' statement of Kuratowski's theorem.

A significant part of the paper was dedicated to proving the 'if' statement of Kuratowski's theorem. To summarize, we first proved a special case by showing that the claim holds for all 3-connected graphs without a Kuratowski subgraph. This result was subsequently extended to include any graph without a Kuratowski subgraph.

We also investigated the question whether a graph can be embedded without intersections on two other surfaces, such as a sphere or a torus. More concretely, we have seen that similar results hold on a sphere whereas the same cannot be said for the torus. In the case of the sphere, properties of the stereographic projection were used to show that planar graphs are equivalent to spherical graphs. For the torus, however, we were able to find an embedding of $K_{3,3}$ and K_5 such that no edges intersect. This is a rather surprising result. What is even more surprising is that the number of forbidden graphs for torus not yet known. However, the Robertson-Seymour theorem assures that this quantity is finite. For more information about this open problem, see [10].

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